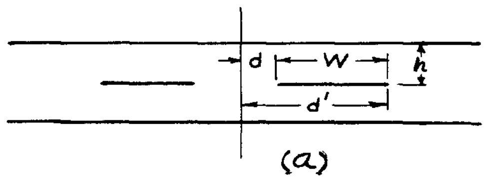
Planar Transmission Lines*

DAVID PARK†

Summary—This paper derives formulas for the transmission properties—characteristic impedance and attenuation—in the principal mode of a transmission line consisting of one or two long strips of metal foil embedded in a dielectric material between two long metal strips considerably wider than the central ones. The width and spacing of the central strips is arbitrary, and it is also necessary to take account of their thickness in computing the attenuation. A graphical method is given for evaluating the characteristic impedance in general, and analytic approximations are given for a number of special cases. Finally the question of the leakage of power from between the outer strips is considered briefly.

INTRODUCTION

ATTENTION has been drawn recently to the possibility of constructing a transmission line in the form of a sandwich, consisting of one or two central conductors of metal foil between slabs of dielectric, the whole inclosed by two broad plates, as shown in cross section in Fig. 1. Where there are two strips the voltage is applied between them and the top and bottom plates are grounded, whereas when there is one, the strip will be one side of the line and the two plates will be the other. We shall assume that the strips are very thin.



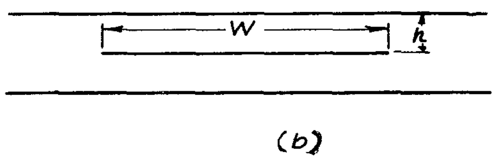


Fig. 1—Cross sections of two transmission lines.

A compact formalism for performing the required analysis in the complex plane has recently been given by Assadourian and Rimai,[1] to which the reader is referred for further details. We shall determine Z_0, the characteristic impedance, in terms of C, the capacitance per unit length of the line, by the relation, valid in the TEM mode,

$$Z_0 = \sqrt{(\mu\epsilon)}/C. \tag{1}$$

(We use rationalized MKS units throughout.) The attenuation in the conductors is determined by P_c, the power dissipated in them per unit length:

$$\alpha = P_c/2P, \tag{2}$$

where

$$P_c = \frac{1}{2}\eta\frac{\epsilon}{\mu}\int |E|^2 ds, \qquad \eta = \sqrt{(\pi f\mu/\sigma_c)} \tag{3}$$

and the integration is over the boundaries of all the conductors; f is the frequency, and σ_c is the conductivity. P represents the power flowing in the line, and is equal to

$$P = V^2/(2Z_0), \tag{4}$$

where V is the maximum instantaneous voltage across the line. We shall not consider loss in the dielectric, though it is easy to do so.[1]

In order to perform the calculations we shall carry out a conformal mapping of the z-plane, with coordinates x and y ($z=x+jy$), which is the plane of Fig. 1, onto the w-plane, with $w=u+jv=f(z)$. (The first step in this mapping is shown in Fig. 2, opposite.) In the w-plane, the lines of constant u are lines of force and those of constant v are equipotentials. It is clear that Δq, the charge per unit length contained between two points on a conductor, is given in terms of the corresponding difference in v by

$$\Delta q = \epsilon\Delta v. \tag{5}$$

To compute the attenuation we need the field strength E, which in the z-plane is

$$E = -\overline{dw/dz}, \tag{6}$$

where the bar denotes the complex conjugate. Then the integral in (3) becomes

$$J = \int_C \left|\frac{dw}{dz}\right|^2 |dz|, \tag{7}$$

where $|dz| = \sqrt{(dx^2+dy^2)}$ is an element of the boundary curve C of the conductor, and this can be written as

$$J = \int_{C'} \left|\frac{dw}{dz}\right| |dw|, \tag{8}$$

where $|dw| = \sqrt{(du^2+dv^2)}$ is an element of the image curve C'.

THE CONFORMAL TRANSFORMATION

We shall first consider the two-strip arrangement of Fig. 1(a); that of Fig. 1(b) follows easily by letting d become infinite. A transformation corresponding to such an arrangement without the bounding planes was

* Supported by the Sprague Electric Co., North Adams, Mass.

† Williams College, Williamstown, Mass.

[1] F. Assadourian and E. Rimai, "Simplified theory of microstrip transmission systems," PROC. IRE, vol. 40, pp. 1651–1657; December, 1952.

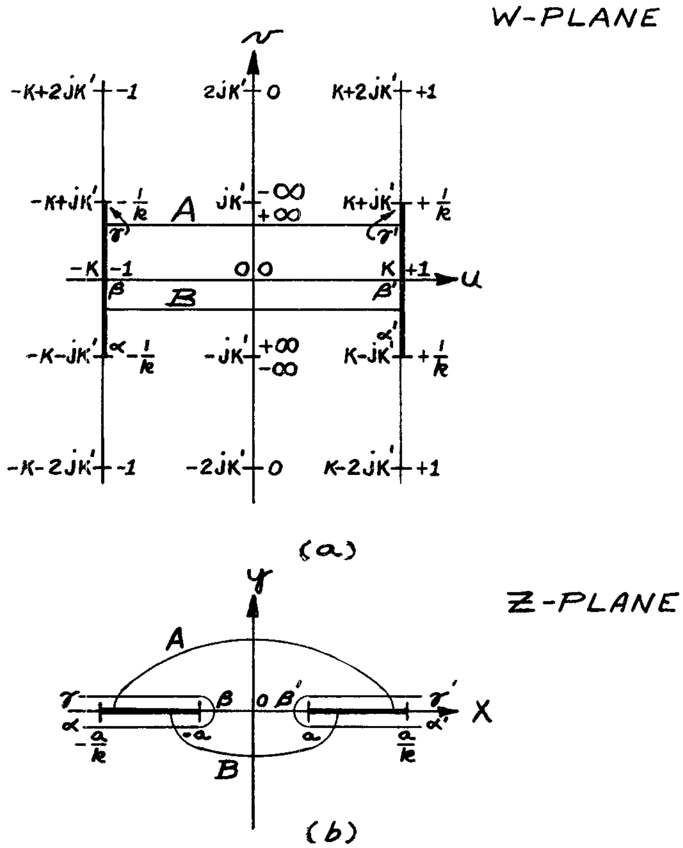


Fig. 2—The Conformal Transformation. In the w-plane, numbers to the left of points give the values of $u+jv$, numbers to the right give the corresponding values of sn w. Points labeled by Greek letters on the two diagrams correspond.

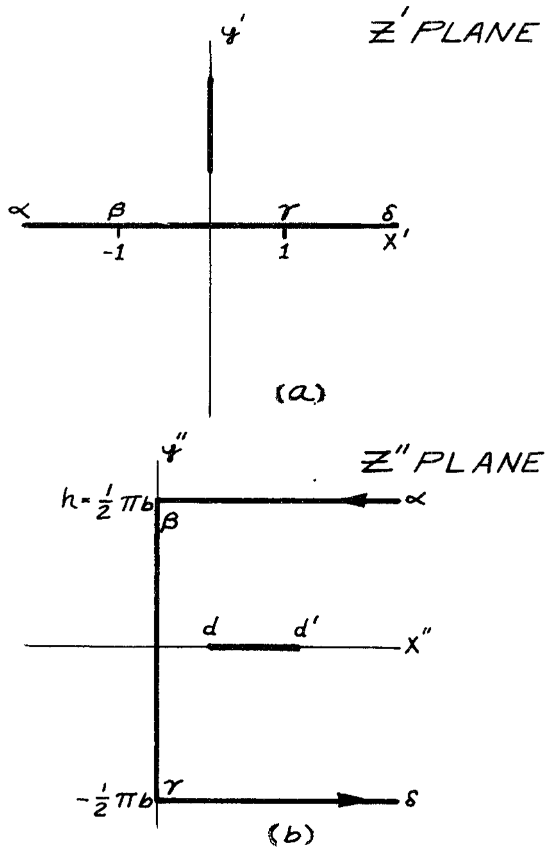


Fig. 3—The second mapping. Points labeled by Greek letters correspond.

given long ago:[2]

$$z = a \operatorname{sn} w \tag{9}$$

where sn w is the Jacobian elliptic function.[3] The maping is shown in Fig. 2. The equipotentials and lines of force form a rectangular grid in the w-plane where the two sides of the right-hand strip in the z-plane are mapped into the single line in the w-plane running from $(K, -jK')$ to $(K, +jK')$, and similarly for the other strip. Two lines of force, A and B, have been drawn on the left and their images given on the right. K and K' are the complete elliptic integrals of the first kind, formed with the complementary moduli k and k' respectively, where

$$k^2 + k'^2 = 1, \tag{10}$$

and k is determined (cf. Fig. 2) by the width of the strip. The total charge per unit length on one strip is ϵ times the difference between the values of v at the beginning and end of the image of the strip—clearly it is $2\epsilon K'$. The potential difference is the difference in the values of u belonging to the two strips, or $2K$. Thus, the capacitance per unit length is

$$C = \epsilon \frac{K'}{K} \tag{11}$$

and the characteristic impedance, by (1), is

$$Z_0 = \sqrt{\left(\frac{\mu}{\epsilon}\right)} \frac{K}{K'} \cdot \tag{12}$$

But this is not the transmission line we want. We still have to introduce the top and bottom plates, and this can be done either by carrying out a new mapping or by using Maxwell's method of images.[4] To do it the first way, let us write instead of (9)

$$z' = jz = ja \operatorname{sn} w, \tag{13}$$

which rotates the z-plane counterclockwise by 90 degrees, but in Fig. 3(a) we draw the configuration in the z'-plane as a single strip and an infinite grounded plane, the second strip now being merely the image of the first. A simple Schwarz-Christoffel transformation

$$z'' = b\left(\cosh^{-1} z' - \frac{1}{2} j\pi\right) \tag{14}$$

now maps the upper half z'-plane into the inside of the

[2] J. J. Thomson, "Recent Researches in Electricity and Magnetism," The Clarendon Press, Oxford, Eng., p. 237; 1892.

[3] A convenient résumé of these functions will be found in R. S. Burington and C. C. Torrance, "Higher Mathematics," McGraw-Hill Book Co., Inc., New York, N. Y.; 1930.

[4] "Treatise on Electricity and Magnetism," Oxford University Press, New York, N. Y., 3rd ed., p. 310; 1904–1946.

polygon of Fig. 3(b). It also maps the strip as shown, and this arrangement is clearly exactly equivalent to that of Fig. 1(a), so the problem is solved. From (14),

$$\sinh\frac{z''}{b} = -jz'$$

or by (13)

$$\sinh\frac{z''}{b} = a \operatorname{sn} w. \tag{15}$$

Comparing Fig. 3(b) with Fig. 1(a), we have by (9) and (15)

$$a = \sinh\frac{d}{b}, \qquad \frac{a}{k} = \sinh\frac{d'}{y}, \qquad h = \frac{1}{2}\pi b \tag{16}$$

so that we can determine k from

$$k = \frac{\sinh d/b}{\sinh d'/b}, \tag{17}$$

with which the solution is formally completed.

The Characteristic Impedance

It will often be convenient in what follows to work with the complementary modulus k', given by (17) as

$$k'^2 = 1 - k^2 = \frac{\sinh\frac{w}{b}\sinh\frac{d+d'}{b}}{\sinh^2\frac{d'}{b}}. \tag{18}$$

Making use of the complementarity of K and K', we can now write (12) as

$$Z_0 = \sqrt{\left(\frac{\mu}{\epsilon}\right)}\frac{K'}{K} \qquad (k'), \tag{19}$$

where the sign on the right means that we are to take k' as the modulus throughout. The quotient, however, is known from the theory of ϑ-functions[5] to be given by

$$\frac{K'}{K} = -\frac{1}{\pi}\ln q = \frac{1}{\pi}f(k^2) \qquad (k),$$

where q is given[6] by

$$q = \frac{1}{2^4}k^2 + \frac{1}{2^5}k^4 + \frac{21}{2^{10}}k^6 + \cdots$$

so that, taking the logarithm of this,[7]

$$\frac{K'}{K} = \frac{1}{\pi}\left(\ln\frac{16}{k^2} - \frac{1}{2}k^2 - \frac{13}{64}k^4 - \frac{23}{192}k^6 - \cdots\right). \tag{20}$$

[5] E. T. Whittaker and G. N. Watson, "Modern Analysis," Cambridge University Press, Cambridge, Eng., 4th ed.; 1927. The proof of this formula, given as a problem on p. 479, follows immediately from the definitions above it.

[6] E. T. Whittaker and G. N. Watson, *ibid.* This follows at once from p. 486. The function $q(k)$ is tabulated and plotted in Jahnke-Emde, "Tables of Functions," Dover Publications, New York, N. Y., ch. IV; 1943.

[7] The series is given in the "Encyklopädie der mathematischen Wissenschaften," vol. II, p. 293.

In Fig. 4 we have plotted $\ln 1/q = f(k^2)$ from the data in the Jahnke-Emde tables, and from (12) and (19) we now have two formulas for Z_0 in ohms:

$$Z_0 = \frac{120 f(k'^2)}{\sqrt{\kappa}} = \frac{120\pi^2}{f(k^2)\sqrt{\kappa}}, \tag{21}$$

where we have set $\sqrt{(\mu/\epsilon)}$ equal to $120\pi/\sqrt{\kappa}$, κ being the dielectric constant of the central material. (The modification of this and following formulas required when the dielectric has a magnetic permeability different from that of free space consists in writing $\mu\kappa/\mu_0$ instead of κ throughout.) In general, it is most convenient to find k' from the dimensions of the line and then read Z_0 off the plot of Fig. 4. In special cases, however, the

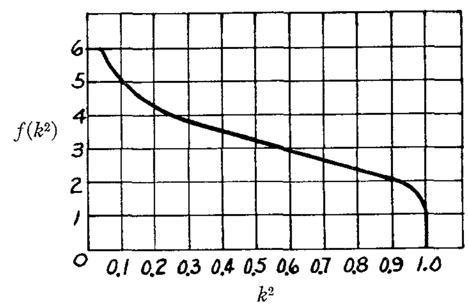


Fig. 4—Plot of $f(k^2) = \ln 1/q$ against k^2. Note that $f(k^2) = \pi^2/f(k'^2)$.

expansion (20) can be used to advantage. Consider for example what happens when the strips are very narrow, so that $w \ll b$.[8] From (18) we have

$$k'^2 \approx 2\frac{w}{b}\coth\frac{d}{b},$$

so that the first term of (20) gives [with (16)]

$$Z_0 \approx \frac{120}{\sqrt{\kappa}}\ln\left(\frac{16h}{\pi w}\tanh\frac{\pi d}{2h}\right) \qquad (w \ll h).$$

As a second example, suppose that the strips are close together ($d \ll b$) but rather wide ($d' \gg b$). Eq. (17) gives

$$k \approx \frac{d}{b}e^{-d'/b} \quad (d \ll b \ll d') \tag{22a}$$

so that again the logarithm suffices, and we have

$$Z_0 \approx \frac{60\pi^2/\sqrt{\kappa}}{\frac{\pi d'}{2h} + \ln\frac{8h}{\pi d}} \quad (d \ll h \ll d'). \tag{22b}$$

As a final special case, let us consider the one-strip line of Fig. 1(b). In (17) we let $d \to \infty$, so that

$$k = e^{-w/b}. \tag{23a}$$

[8] There will be a number of these "double inequalities" in what follows. Here, $w \ll b$ means that w/b is small enough so that higher terms in it can be neglected. Since in nearly every case ratios of this kind appear in exponential or hyperbolic expressions, it usually suffices for purposes of ordinary accuracy to read "$a \ll b$" as "$a < \frac{1}{2}b$" or "$a \gg b$" as "$a > 2b$."

In this line the potential difference is only half that in the case of two strips, so that Z_0 is given by half its value in (21). In the extreme cases we have as before

$$Z_0 \approx \frac{60}{\sqrt{\kappa}} \ln \frac{16h}{\pi w} (w \ll h), \quad Z_0 \approx \frac{15\pi^2/\sqrt{\kappa}}{\ln 2 + \frac{\pi w}{4h}} (w \gg h). \quad (23b)$$

The sign $\gg$ in the second formula is actually unnecessary, for the error is already very small when $w = h$.[9]

The Attenuation

In order to calculate the attenuation from (8) we need to know dw/dz. From (15) (dropping the now unnecessary primes) we find, in terms of z and w,

$$\frac{dw}{dz} = \frac{a}{b} \frac{\cosh z/b}{\sqrt{\left[\left(a^2 - \sinh^2 \frac{z}{b}\right)\left(a^2 - k^2 \sinh^2 \frac{z}{b}\right)\right]}} \quad (24a)$$

$$= \frac{1}{ab} \frac{\sqrt{(1 + a^2 \operatorname{sn}^2 w)}}{\operatorname{cn} w \operatorname{dn} w}. \quad (24b)$$

The second of these is not hard to integrate. Let us begin by calculating J_1, the contribution to the dissipation integral (8) due to the top and bottom plates. The contour in (8) corresponding to half the top plate, is by Fig. 3, that for which $-\infty < z' \leq -1$. On this plate $u = 0$, so that $w = jv$, $\operatorname{sn} w = j \operatorname{tn}(v, k)$, and (13) becomes[3]

$$z' = -a \operatorname{tn} v \quad (k'),$$

so that (8) is

$$\frac{1}{4} J_1 = \frac{1}{ab} \int_{tn^{-1} 1/a}^{K} \frac{\sqrt{|1 - a^2 \operatorname{tn}^2 v|}}{\operatorname{dn} v} \operatorname{cn}^2 v dv \quad (k').$$

This is evaluated by letting $s = \sqrt{(1 + a^2)} \operatorname{sn} v$ to give

$$J_1 = \frac{4}{ab\sqrt{(1+a^2)}} \int_1^{\sqrt{(1+a^2)}} \frac{\sqrt{(s^2 - 1)}}{1 - k''^2 s^2} ds \quad \left(k''^2 = \frac{k'^2}{1 + a^2}\right) \quad (25)$$

$$= \frac{4\sqrt{(1+a^2)}}{abk'^2} \left[\sqrt{\left(\frac{k^2 + a^2}{1 + a^2}\right)} \tanh^{-1} \frac{a}{\sqrt{(k^2 + a^2)}} - \tanh^{-1} \frac{a}{\sqrt{(1 + a^2)}}\right]$$

$$= \frac{4}{b^2 k'^2} \left(d' \coth \frac{d'}{b} - d \coth \frac{d}{b}\right). \quad (26a)$$

When k' is close to zero this form is awkward, but then we can do (25) directly to get

$$J_1 \approx \frac{2}{b}\left(1 - \frac{2d/b}{\sinh 2d/b}\right) \quad (k' \ll 1) \quad (26b)$$

and similarly the limiting form when d and d' are large is

$$J_1 \approx \frac{4w}{b^2 k'^2} \quad (d \gg b). \quad (26c)$$

[9] These formulas have been given by R. M. Barrett in "Microwave Printed Circuits—Preliminary Memo and Technical Note," AF Research Center, Cambridge, Mass.; 1951, together with the results of extensive experiments verifying them.

To find the power absorbed in the two strips we must, for the first time in this work, introduce the thickness of the strips, for if we take it to vanish, the field intensity, and hence the dissipation, will be infinite at the edges. To avoid further complexity we shall assume that the strip follows the equipotential $v = K - \epsilon$, where $\epsilon/K \ll 1$. From (24a) we have, evaluated along the strip, with $dw = \epsilon$ and $dz = jdy = j\lambda$,

$$J \frac{dw}{dz} = \frac{\epsilon}{\lambda} = \frac{a \cosh x/b}{\sqrt{\left[\left(\sinh^2 \frac{x}{b} - a^2\right)\left(a^2 - k^2 \sinh^2 \frac{x}{b}\right)\right]}}.$$

To find the approximate half-thickness of this surface, we write λ as a function of ϵ and x, and then find its maximum value. This turns out to be

$$\lambda_{\max} = b\epsilon \frac{\cosh d'/b - \cosh d/b}{\sinh d'/b}$$

so that if we take t to be the (total) thickness of the strip, the corresponding equipotential surface has

$$\epsilon = \frac{t}{2b} \frac{\sinh d'/b}{\cosh d'/b - \cosh d/b}. \quad (27)$$

This artifice of introducing the strip by means of an equipotential amounts to replacing square edges by rounded ones. An actual strip of foil, however, has edges which are neither square nor round, and the point is of little importance.

Now we are to integrate (24b). For the top side of one strip we integrate along the equipotential $w = K + \epsilon + jv$ $(0 \leq v \leq K')$. Making use of the approximations for small ϵ we have, to sufficient accuracy,

$$\left.\begin{aligned} \operatorname{sn}(K + \epsilon + jv) &\approx \frac{1}{\operatorname{dn} v} - j\epsilon k^2 \frac{\operatorname{sn} v \operatorname{cn} v}{\operatorname{dn}^2 v} \\ \operatorname{cn}(K + \epsilon + jv) &\approx -jk\left(\frac{\operatorname{sn} v}{\operatorname{dn} v} - j\epsilon \frac{\operatorname{cn} v}{\operatorname{dn}^2 v}\right) \\ dn(K + \epsilon + jv) &\approx k\left(\frac{\operatorname{cn} v}{\operatorname{dn} v} + j\epsilon k'^2 \frac{\operatorname{sn} v}{\operatorname{dn} v}\right) \end{aligned}\right\} \quad (k').$$

With this, the contribution to (8) from one strip becomes

$$J_2 = \frac{2}{abk^2} \int_0^K \frac{\sqrt{(a^2 + \operatorname{dn}^2 v)} \operatorname{dn} v dv}{\operatorname{sn} v \operatorname{cn} v \left[1 + \frac{\epsilon^2}{\operatorname{dn}^2 v}\left(\frac{\operatorname{cn} v}{\operatorname{sn} v} - k'^2 \frac{\operatorname{sn} v}{\operatorname{cn} v}\right)^2\right]} \quad (k')$$

for the dissipation in one side of one strip. The substitution $s = \operatorname{dn}^2 v$ now reduces this to

$$J_2 = \frac{1}{ab} \int_{k^2}^{1} \frac{\sqrt{(a^2 + s)} ds}{(1 - s)(s - k^2) + \frac{\epsilon^2}{s}[(1 + k^2)s^2 - 2k^2]^2} \quad (k)$$

The result of doing this integration, dropping all terms which vanish with ϵ, is, by (27),

$$J_2 = \frac{2}{bk'^2}\left[\coth\frac{d}{b}\ln\left(\frac{4b}{t}\cosh\frac{d}{b}\frac{\sinh w/2b}{\cosh d''/2b}\right) + \coth\frac{d'}{b}\ln\left(\frac{4b}{t}k\cosh\frac{d'}{b}\frac{\sinh w/2b}{\cosh d''/2b}\right)\right] \tag{28}$$

where d'' is $d'+d$. This can be evaluated readily enough in any particular instance. If we assume that d'/b and d/b are large enough so that $e^{-2d/b}$ and $e^{-3d/b}$ can be neglected, it simplifies considerably, and we find that

$$J_2 \approx \frac{4}{bk'^2}(1 + e^{-2d/b})\left[\ln\left(\frac{4b}{t}\sinh\frac{w}{2b}\right) - \frac{w}{2b}\right] \tag{29}$$

and that for the two conductors

$$J = J_1 + 2J_2 \approx \frac{8}{bk'^2}\left[(1 + e^{-2d/b})\ln\left(\frac{4b}{t}\sinh\frac{w}{2b}\right) - \frac{w+2d}{2b}e^{-2d/b}\right]. \tag{30}$$

Here we shall give only two further limiting forms. The first is that corresponding to (22a) where, for the whole system, (8) is

$$J = J_1 + 2J_2 \approx \frac{4}{d}\left[\ln\frac{4b}{t} + \frac{d}{b^2}(w - 2b)\right] \quad (d \ll b \ll d')$$

so that, with (2), (3), (4) and (22b), the attenuation is

$$\alpha \approx \frac{\eta\sqrt{\kappa}}{120\pi^2 d}\,\frac{\ln\frac{8h}{\pi t} + \frac{\pi^2 d}{4h^2}\left(w - \frac{4}{\pi}h\right)}{\ln\frac{8h}{\pi d} + \frac{\pi d'}{2h}} \quad (d \ll b \ll d')$$

where we have used the approximation

$$K \approx \frac{\pi}{2} \quad (k \ll 1).$$

The other limiting case which we shall consider is that in which d becomes infinite, corresponding to the single-strip line:

$$J_2 \approx \frac{4}{bk'^2}\left(\ln\frac{2b}{t} - e^{-w/b}\right) \quad (w \gg b). \tag{31}$$

In these formulas, t must here be taken small enough so that the second term in the numerator is not larger than (or nearly equal to) the first. To find the total dissipation in this case we use (26c) and (29) to form

$$J = J_1 + J_2 = \frac{4}{bk'^2}\left(\ln\frac{2b}{t} + \frac{w}{b} - e^{-w/b}\right) \quad (w \gg b). \tag{32}$$

This, with (2), (3), (4), and (23b) gives for the attenuation in this case.

$$\alpha \approx \frac{\eta\sqrt{\kappa}}{120\pi h}\,\frac{\ln\frac{4h}{\pi t} + \frac{\pi w}{2h}}{\ln 2 + \frac{\pi w}{4h}} \quad (w \gg b). \tag{33}$$

Leakage from the Line

One way to reduce the leakage of power from the edges of the plates is to close them in. This, of course, changes somewhat the electrical characteristics of the line, but the chief objection to it lies in the increased complexity of the operation of making the "sandwich." Although to calculate exactly the power which leaks from the line is beyond the scope of this paper, we can at least make a relative estimate of it by calculating the intensity of the electric flux at points in the central plane of the system. One would at once think that where there are two strips with opposite charge the lines of force would predominantly run from one to the other, so that there would be less flux out to the sides than in the single-strip line. We shall see that this is true, though the effect is not so pronounced as one might have thought.

If we square (24a) and let z be a real number x, much greater than b, we find that

$$E^2 \approx \frac{4a^2}{k^2b^2}e^{-2x/b}. \tag{34}$$

But throughout this analysis we have assumed that the voltage applied across the line is given by K, which depends on the line geometry. What is relevant here is, say, the field produced when one volt is applied. From (34), this is

$$E_1^2 \approx \frac{4a^2}{K^2k^2b^2}e^{-2x/b}.$$

For the double-strip line this is, by (16)

$$E_1^2 \approx \frac{4}{K^2b^2}\sinh^2\frac{d'}{b}e^{-2x/b} \quad \text{(double strip)}. \tag{35}$$

For the single-strip line x should be measured from the center of the strip. Therefore we replace x in (34) by $x+d+\frac{1}{2}w$, and further, the potential difference between the strip and the envelope is only half that between the two strips. Thus

$$E_1^2 \approx \frac{4}{K^2b^2}e^{-2(x-w/2)/b} \quad \text{(single strip)}. \tag{36}$$

Supposing that, as would generally be the case, $d' \gg b$ in (35), we find that

$$E_1^2 \approx \frac{1}{K^2b^2}e^{-2(x-d')/b} \quad \text{(double strip)}. \tag{37}$$

These two special cases will in general have $k^2 \ll 1$ [see (22a) and (23a)]. If this is true then K is close to $\pi/2$ in both cases. Further, comparing (37) with (36) we see that in each the factor in parentheses is the distance measured from the outside of a strip, so that if the one or two strips are to occupy a given width, the two-strip arrangement will, for a given width of plate, have one-fourth the leakage of the one-strip system. These equations also make it clear why the most compact and economical construction will have the dielectric sheets made as thin as possible.